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ON OPTIMAL STABILIZATION OF THE EQUILIBRIUM POSITIONS OF A SOLID WITH A CAVITY CONTAINING A FLUID

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Results obtained in [1] are extended to embrance the problem of optimal stabilization of the position of equilibrium of a solid with a cavity containing a homogeneous, viscous incompressible fluid, with respect to a part of the generalized coordinates, generalized velocities and kinetic energy of the fluid.

1. Let us consider a solid with a singly-connected cavity partly or completely filled with a homogeneous, viscous incompressible fluid. Let $q_1, \ldots, q_n$ $(n \leqslant 6)$ denote the generalized coordinates of the system. We assume that the constraints imposed on the system are time independent, and the system is acted upon by potential forces as well as certain additional forces of the type [2]

$$Q_i = \sum_{j=1}^{r} m_{ij}(\mathbf{q})\, w_j(\mathbf{q}, \mathbf{q}^{\cdot}) \tag{1.1}$$

where w_j denote the control functions. The surface tension is neglected. The equations of motion are written in the form [3]

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = \Phi_i + \sum_{j=1}^{r} m_{ij} w_j \quad (i = 1, \ldots, n \leqslant 6) \tag{1.2}$$

The latter equations should be supplmented by the Navier–Stokes and continuity equations, and the corresponding boundary and initial conditions.

Using the total energy of the system $H = T + U$ as the Liapunov function, we obtain

$$H^{\cdot} = -\int_{\tau} E\, d\tau + \sum_{i=1}^{n}\sum_{j=1}^{r} m_{ij} w_j q_i^{\cdot} \tag{1.3}$$

$$E = 2\mu \sum_{i,j=1}^{3} e_{ij}^2, \quad e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

Here $\mathbf{v}$ (v_1, v_2, v_3) is the fluid velocity vector relative to the fixed coordinate system; $x_1x_2x_3$ is the moving coordinate system and in what follows $\mathbf{u}$ (u_1, u_2, u_3) denotes the fluid velocity vector in the latter system.

We shall assume that [1, 4]

1) the equations of motion (1.2) admit, at $w_j = 0$, a particular solution $\mathbf{q} = \mathbf{q}^{\cdot}$

$= 0$, $\mathbf{v} = 0$ (the position of equilibrium);

2) the potential energy U is positive-definite with respect to $q_1, \ldots q_m$ ($m < n$); moreover, by virtue of (1.3) the position of equilibrium is stable with respect to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}, T_2$ when $w_j = 0$ (T_2 is the kinetic energy of the fluid);

3) the coordinates $q_{m+1}, \ldots, q_n$ are angular (mod 2π), and all quantities appearing in (1.2) as well as the function H, are 2π-periodic in $q_{m+1}, \ldots, q_n$. It can also be assumed that $q_{m+1}(t), \ldots, q_n(t)$ are bounded when the motion is perturbed [4, 5];

4) when $w_j = 0$, there are no positions of equilibrium in the set $U > 0$.

Following [2], we formulate the problem (compare with [1]) of determining the controls $w_j = w_j^\circ$ ensuring the asymptotic stability of the positions of equlibrium $\mathbf{q} = \mathbf{q}^{\cdot} = 0, \mathbf{v} = 0$ with respect to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}, T_2$ and minimizing the functional

$$J = \int_0^\infty \Big(\psi + \sum_{i,j=1}^r \beta_{ij} w_i w_j\Big) dt \tag{1.4}$$

in which ψ is a non-negative function to be determined and the quadratic form is a positive-definite function of the controls.

We regard, as class $K = \{\mathbf{w}(\mathbf{q}, \mathbf{q}^{\cdot})\}$ of control functions $\mathbf{w}(\mathbf{q}, \mathbf{q}^{\cdot})$ the set of continuous functions of $\mathbf{w}(\mathbf{q}, \mathbf{q}^{\cdot})$ satisfying the condition [1]

$$w_j = 0 \quad \text{when } q_1 = \ldots = q_m = q_1^{\cdot} = \ldots = q_n^{\cdot} = 0 \quad (j = 1, \ldots, r) \tag{1.5}$$

Further, as in [4, 6], we shall introduce certain assumptions concerning the character of the perturbed motions.

Let us perform a continuous change of variables [6]

$$\lambda = \lambda(x_1, x_2, x_3), \quad \nu = \nu(x_1, x_2, x_3), \quad \tau = W(x_1, x_2, x_3)$$

write the equation of the side wall as $\nu(x_1, x_2, x_3) = \beta_0 = \text{const}$ and let the equation of free surface of the fluid be represented, for every $\mathbf{w}(\mathbf{q}, \mathbf{q}^{\cdot}) \in K$, in the form

$$\tau - \alpha_0 = \varkappa(t, \lambda, \nu), \quad \alpha_0 = \text{const}.$$

We assume that by specifying, for every $\mathbf{w}(\mathbf{q}, \mathbf{q}^{\cdot}) \in K$, the following quantities at the initial moment $t = t_0$

$$\mathbf{q}_0, \quad \mathbf{q}_0^{\cdot}, \quad u_i(t_0, x_1, x_2, x_3) = \varphi_i(x_1, x_2, x_3) \quad (i = 1, 2, 3), \quad \varkappa(t_0, \lambda, \nu)$$

with $\operatorname{div} \mathbf{u} = 0$, the subsequent motion of the system is determined uniquely.

Let the deviation ∇ in every perturbed motion at any $t \geqslant t_0 \geqslant 0$ satisfy the condition [3] $\nabla > \varepsilon l$, and let the following assumptions hold for every $\mathbf{w}(\mathbf{q}, \mathbf{q}^{\cdot}) \in K$:

A. [4] In each perturbed motion

$$\|\mathbf{u}\| \leqslant M, \quad \|\mathbf{u}^{\cdot}\| \leqslant M, \quad |e_{ij}| \leqslant M, \quad |e_{ij}^{\cdot}| \leqslant M$$

$$|\partial e_{ij} / \partial x_s| \leqslant M, \quad |\partial u_i / \partial x_j| \leqslant M \quad (s, j, s = 1, 2, 3; M = \text{const})$$

B. [4, 6] The function $\varkappa(t, \lambda, \nu)$ is continuous in λ and ν uniformly in $t \geqslant 0$, i.e. for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|\lambda' - \lambda''| < \delta$, $|\nu' - \nu''| < \delta$ implies $|\varkappa(t, \lambda', \nu) - \varkappa(t, \lambda'', \nu'')| < \varepsilon$ for all $t \geqslant 0$.

C. [4, 6] Function H depends continuously on the initial conditions, i.e. for every $\varepsilon > 0, \theta > 0$ there exists $\delta(\varepsilon, \theta) > 0$ such that

$$\|\mathbf{q}_0' - \mathbf{q}_0''\| < \delta, \quad \|\mathbf{q}_0'^{\cdot} - \mathbf{q}_0''^{\cdot}\| < \delta, \quad |\varphi_i'(x_1, x_2, x_3) - \varphi_i''(x_1, x_2, x_3)| < \delta$$
$$|\varkappa'(0, \lambda, \nu) - \varkappa''(0, \lambda, \nu)| < \delta$$

implies

$$|H(\mathbf{q}'[\theta], \mathbf{q}'^{\cdot}[\theta], \mathbf{u}'[\theta], \varkappa'[\theta]) - H(\mathbf{q}''[\theta], \mathbf{q}''^{\cdot}[\theta], \mathbf{u}''[\theta], \varkappa''[\theta])| < \varepsilon$$

Let us consider the expression [7] $B[H, \mathbf{q}, \mathbf{q}^{\cdot}, \mathbf{v}, \mathbf{w}] = H^{\cdot} + \omega$ in which ω denotes the integrand function of (1.4) and the function $H^{\cdot}$ is determined by (1.3). Using the conditions $B[H, \mathbf{q}, \mathbf{q}^{\cdot}, \mathbf{v}, \mathbf{w}^\circ] = 0$ and $B = [H, \mathbf{q}, \mathbf{q}^{\cdot}, \mathbf{v}, \mathbf{w}] \geqslant 0$ for all $\mathbf{w} \in K$ we can show, as in [2], that the optimal controls w_j° and the function ψ have the form (Δ_{kj} is the algebraic complement of the element β_{kj})

$$w_j^\circ = -\frac{1}{2}\sum_{k=1}^{r}\frac{\Delta_{kj}}{\Delta}\sum_{i=1}^{n} m_{ik} q_i^{\cdot} \quad (\Delta = \det\|\beta_{ij}\|) \tag{1.6}$$

$$\psi = \int_\tau E\, d\tau + S, \qquad S = \sum_{i, j=1}^{r} \beta_{ij} w_i^\circ w_j^\circ \tag{1.7}$$

Let us assume that the quadratic form S is positive-definite with respect to $q_1^{\cdot}, \ldots, q_n^{\cdot}$. Taking into account the fact that (compare with [2])

$$H^{\cdot}|_{\mathbf{w}=\mathbf{w}^\circ} = -\int_\tau E\, d\tau - 2S$$

we conclude, with the help of [4], that the position of equilibrium $\mathbf{q} = \mathbf{q}^{\cdot} = 0$, $\mathbf{v} = 0$ with $w_j = w_j^\circ$, asymptotically stable with respect to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}, T_2$ and $\lim H(q^\circ[t], q^{\cdot\circ}[t], \mathbf{u}^\circ[t], \varkappa^\circ[t]) = 0$ as $t \to \infty$.

N o t e. Since E is a positive-definite quadratic form of the deformation rate tensor components, the first term in the formula for ψ (see (1.7))plays, with respect to the fluid, a part analogous to that of S with respect to the generalized velocities. Thus the function ψ in (1.4) characterizes the decay rate of both generalized velocities and of the relative motion of the fluid.

Let now $w_j^* \in K$ denote any control ensuring the asymptotic stability of the equilibrium $\mathbf{q} = \mathbf{q}^{\cdot} = 0, \mathbf{v} = 0$ relative to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}, T_2$. We shall show that

$$\lim_{t\to\infty} H(q^*[t], q^{\cdot *}[t], \mathbf{u}^*[t], \varkappa^*[t]) = 0 \tag{1.8}$$

Assume the opposite. The sequence of functions $\{\varkappa^*[t_s, \lambda, \nu]\}$ is uniformly bounded and has the same order of continuity (see assumption B) for any sequence $t_s \to \infty$, therefore by virtue of the Arzell theorem we can separate from it a convergent subsequence. Thus by virtue of the conditions 3) and (1.5), the following relations hold for some sequence $t_k \to \infty$:

$$\mathbf{q}^*[t_k] \to \mathbf{q}_*, \quad \mathbf{q}^{\cdot *}[t_k] \to 0, \quad \mathbf{u}^*[t_k] \to 0, \quad \varkappa^*[t_k, \lambda, \nu] \to \varkappa_*(\lambda, \nu), \quad \mathbf{w}^*[t_k] \to 0$$

Clearly the point $(\mathbf{q}_*, \mathbf{q}^{\cdot} = 0, \mathbf{u} = 0, \varkappa = \varkappa_*, \mathbf{w} = 0)$ is a position of equilibrium. $T = 0$ at this point, therefore if the relation (1.8) does not hold, then $U > 0$ at this point and this contradicts the assumption 4). Applying now Theorem 2 of [1], we arrive at the following conclusion: the controls (1.6) solve the problem of optimal stabilization, with respect to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}, T_2$, of the position of equilibrium $\mathbf{q} = \mathbf{q}^{\cdot} = 0, \ \mathbf{v} = 0$, under the criterion of control quality (1.4), (1.7).

N o t e s. 1) The system in question has an infinite number of degrees of freedom. The proof of Theorem 2 of [1] is nevertheless still retained.

2) The above result remains valid if the quadratic form S is positive-definite with respect to $q_1^{\cdot}, \ldots, q_n^{\cdot}$ and the set [1,8] $(H > 0) \cap (S = 0)$ contains no motion of the whole system as a single rigid body, (necessary and sufficient conditions for such a motion to be present are given in the theorem in [2]).

2. E x a m p l e 1 [9]. Let us consider the problem of optimal stabilization of the motions of a dynamically symmetric satellite during which the mass center of the satellite rotates along a circular orbit about the center of attraction, and the axis of symmetry is perpendicular to the orbital plane of the mass center. We assume that the body of the satellite contains a cavity completely filled with a homogeneous viscous incompressible fluid. We retain the notation and formulation of the problem given in [9]. Let the controlling moment corresponding to the coordinate ψ_1 and minimizing the functional have the form (2.1) and (2.2)

$$Q_{\psi_1} = mw \tag{2.1}$$

$$J = \int_0^\infty (\psi + \beta w^2)\, dt \tag{2.2}$$

According to (1.7) and (1.8), we have

$$w^\circ = -\frac{m}{2\beta}\psi_1^{\cdot} \tag{2.3}$$

$$\psi = \int_\tau E\, d\tau + \frac{m^2}{4\beta}\psi_1^{\cdot 2} \tag{2.4}$$

In the case when the body has no cavity filled with fluid, the results obtained in Sect. 1 coincide with the example given in [1]. To illustrate this we consider the problem of optimal stabilization of motions of a perfectly rigid, dynamically symmetric satellite during which the mass center moves along a circular orbit and the symmetry axis either points towards the center of attraction, or is perpendicular to the orbital plane of the mass center [9].

E x a m p l e, 2. First we investigate the problem of optimal stabilization of the dynamic symmetry axis relative to the center of attraction (see Sect. 1 of [9]). Let the control moments corresponding to the coordinates ψ_1 and ψ_2, and the minimizing functional, have the form

$$Q_{\psi_1} = m_1 w_1, \quad Q_{\psi_2} = m_2 w_2 \tag{2.5}$$

$$J = \int_0^\infty (\psi + \beta^{(1)} w_1^2 + \beta^{(2)} w_2^2)\, dt \tag{2.6}$$

In accordance with [1, 2] (compare with (1.9) and (1.10), we obtain

$$w_1^\circ = -\frac{m_1}{2\beta^{(1)}} \psi_1^{\cdot}, \quad w_2^\circ = -\frac{m_2}{2\beta^{(2)}} \psi_2^{\cdot} \tag{2.7}$$

$$\psi = \frac{m_1^2}{4\beta^{(1)}} \psi_1^{\cdot 2} + \frac{m_2^2}{4\beta^{(2)}} \psi_2^{\cdot 2} \tag{2.8}$$

Next we turn our attention to the problem of optimal stabilization of the dynamic symmetry axis in the direction perpendicular to the oribital plane [9]. Let the control moments corresponding to the coordinates φ_1, φ_2 and ψ_1, and minimizing the functional, have the form

$$Q_{\varphi_1} = m_1 w_1, \quad Q_{\varphi_2} = m_2 w_2, \quad Q_{\psi_1} = m_3 w_3 \tag{2.9}$$

$$J = \int_0^\infty (\psi + \beta^{(1)} w_1^2 + \beta^{(2)} w_2^2 + \beta^{(3)} w_3^2)\, dt \tag{2.10}$$

Moreover we obtain as before

$$w_1^\circ = -\frac{m_1}{2\beta^{(1)}} \varphi_1^{\cdot}, \quad w_2^\circ = -\frac{m_2}{2\beta^{(2)}} \varphi_2^{\cdot}, \quad w_3^\circ = -\frac{m_3}{2\beta^{(3)}} \psi_1^{\cdot} \tag{2.11}$$

$$\psi = \frac{m_1^2}{4\beta^{(1)}} \varphi_1^{\cdot 2} + \frac{m_2^2}{4\beta^{(2)}} \varphi_2^{\cdot 2} + \frac{m_3^2}{4\beta^{(3)}} \psi_1^{\cdot 2} \tag{2.12}$$

Examples 1 and 2 show that the stabilization rules used in [9] in the form of dissipative forces are optimal with respect to the functionals (2.2), (2.4), (2.6), (2.8), (2.10) and (2.12), if $f = m^2 \psi_1^{\cdot 2} / 4\beta$ is used as the Rayleigh function [9] in Example 1, and analogous expressions for f in the problems dealt with in Example 2.

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